Lecture 4

Recall:

polydisk $\qquad\qquad \forall z \in D^n, j:\ \zeta \to u(z_1, \dots, \zeta, \dots, z_n)$ holom. near z_j

Cauchy's Formula in D^n. Let u cont. in $\overline{D^n}$, sep. hol. in D^n. Then

$$u(z) = \frac{1}{(2\pi i)^n} \int_{\partial D} \frac{u(\zeta)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)}\, d\zeta_1 \dots d\zeta_n .$$

$\partial D_1 \times \dots \times \partial D_n$

Consequences.

(1) Theoretical sidetrack:

Hartogs Thm - I. Let u be function $\Omega \subseteq \mathbb{C}^n \to \mathbb{C}$ (no regularity assumed). If u is separately holomorphic in Ω, then u is holomorphic.

Remark: If $u \in C^1$, then this is essentially the def. of holomorphic. If u is cont., then we proved this as cor. of Cauchy's Formula. To prove without any reg. is more difficult and somewhat technical. It's an induction over the dimension and uses the power series expansion (below). We omit the pf and refer to Hörmander.

(2) Max principle: If u is holom. in Ω, $|u|$ cannot achieve max. inside. In fact:

Ex. Let u be holom. in D^n, cont. on $\overline{D^n}$. Then

$$\max_{\overline{D^n}} |u| = \max_{\partial_0 D^n} |u| .$$

Power Series Expansion.

Def. 3. A series $\sum_\alpha a_\alpha(z)$ converges normally in $\Omega \subseteq \mathbb{C}^n$ if

countable sum

$$\sum_\alpha \sup_K |a_\alpha(z)| \text{ converges}, \quad \forall \text{ compacts } K \subset\subset \Omega .$$

$\sum_\alpha \sup_K |a_\alpha(z)|$ converges, $\forall$ compacts $K \subset\subset \Omega$.

Rem. Normal conv. $\Rightarrow$ uniform conv. $\Rightarrow$ $\sum_\alpha a_\alpha(z)$ converges to function $a(z)$. If a_α are cont., then a is cont. If a_α are holo., then a is holo.

Thm 1. If u is holo. in polydisk $D^n \subseteq \mathbb{C}^n$ centered at $a \in \mathbb{C}^n$, then
$$u(z) = \sum_{\alpha \in \mathbb{Z}_+^n} u_\alpha (z-a)^\alpha \ ; \quad u_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha u}{\partial z^\alpha}(a)$$
with normal convergence in D^n.

Note: Here, $(z-a)^\alpha = (z_1-a_1)^{\alpha_1} \dots (z_n-a_n)^{\alpha_n}$ and $\alpha! = \alpha_1! \dots \alpha_n!$.

Pf. Follows from Cauchy's Formula as in 1 variable. ◪

Cauchy's Estimates. Let $D^n \subseteq \mathbb{C}^n$ be polydisk of polyradius r, centered at a. If u is holo. in D^n, and $|u| \le M$ in D^n, then
$$|u_{z^\alpha}(a)| \le \frac{M\alpha!}{r^\alpha}.$$

Pf. Follows from CF as in 1 variable. ◪

Another consequence of the power series expansion (as in $\mathbb{C}$) is unique continuation: If u is holom., $u_{z^\alpha}(a) = 0$ $\forall \alpha \in \mathbb{Z}_+^n$, then $u \equiv 0$ in an open polydisk D^n centered at a. If u holom. in $\Omega \subseteq \mathbb{C}^n$, Ω connected, and $u \equiv 0$ on open polydisk $D^n \subseteq \Omega$, then $u \equiv 0$ in Ω.

$\bar{\partial}$-equation.

Let f be $(0,1)$-form and consider equation for fcn u:

$$\bar{\partial} u = f. \quad \left(= \sum_{j=1}^{n} f_j \, d\bar{z}_j\right).$$

Recall that necessary for solution is $\bar{\partial} f = 0$. As system:

$$\partial_{\bar{z}_j} u = f_j \,, \quad \text{where} \quad \partial_{\bar{z}_k} f_\ell = \partial_{\bar{z}_\ell} f_k .$$

Recall: Given a form f in $\Omega \subseteq \mathbb{C}^n$, the support of v is

$$\operatorname{supp} v = \overline{\{ z \in \Omega : f(z) \neq 0 \}}.$$

$n>1$ ← compact supp. of u fails in $\mathbb{C}$.

Thm1. Let f be $(0,1)$-form in $\mathbb{C}^n$ of class $\mathcal{C}^k$ w/ compact support $\left(f \in \mathcal{C}^k_0(\mathbb{C}^n) \right)$. If $\bar{\partial} f = 0$, then $\exists$ $u \in \mathcal{C}^k_0(\mathbb{C}^n)$ s.t. $\bar{\partial} u = f$.

For pf, we first consider the case $n=1$:

Prop1. Let $f = \varphi \, d\bar{z}$, $\varphi \in \mathcal{C}^k_0(\mathbb{C})$. Then,

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\varphi(\zeta) \, d\zeta}{\zeta - z}$$

is $\mathcal{C}^k$ in $\mathbb{C}$ and solves $\bar{\partial} u = f$.

Pf of Prop1. We recall that $\frac{1}{|\zeta|}$ is integrable over any bdd open set $\Omega \subset\subset \mathbb{C}$. Next, by COV,

$$u(z) = \left\{ \begin{matrix} \zeta - z = \zeta' \\ d\zeta = d\zeta' \end{matrix} \right\} = \frac{1}{2\pi i} \int \frac{\varphi(z+\zeta') \, d\zeta'}{\zeta'} \quad (1)$$

$$u(z) = \left\{ \begin{matrix} \zeta - z = \zeta' \\ d\zeta = d\zeta' \end{matrix} \right\} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\varphi(z+\zeta')\, d\zeta'}{\zeta'}, \quad (1)$$

so we see that u is as regular as φ, i.e., $\mathcal{C}^k$ in this case.

Now, let Ω be $\underbrace{\text{bdd open set}}_{}$ (w/ $\partial\Omega$ smooth) s.t. $\operatorname{supp}\varphi \subset\subset \Omega$.

Then, by General CF:

$$\varphi(z) = \frac{1}{2\pi i}\left(\underbrace{\int_{\partial\Omega} \frac{\varphi(\zeta)}{\zeta - z} d\zeta}_{=0} + \int_{\Omega} \frac{\varphi_{\bar\zeta}(\zeta)}{\zeta - z} d\zeta \right)$$

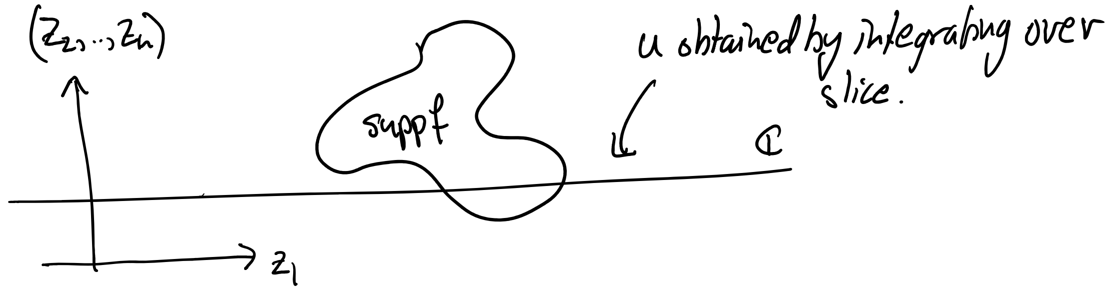
$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\varphi_{\bar\zeta}(\zeta)}{\zeta - z} d\zeta \underset{\text{as above}}{=} \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\varphi_{\bar\zeta}(z+\zeta')}{\zeta'} d\zeta' \quad (2)$$

Thus, applying $\partial_{\bar z}$ to (1) shows $\partial_{\bar z} u = f$. ▨

<u>Pf of Thm1</u>. Consider

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta' + z_1, z_2, \ldots)}{\zeta'} d\zeta'$$

As above, $u \in \mathcal{C}^k$ and, since $n > 1$, $u = 0$ when $|z_2|^2 + \ldots + |z_n|^2$ large:



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$$\partial_{\bar{z}_k} u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial_{\bar{z}_k} f_1(\zeta' + z_1, \dots, z_n)}{\zeta'} d\zeta' = \left\{ \partial_{\bar{z}_k} f_1 = \partial_{\bar{z}_1} f_k \right\}$$

$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial_{\bar{z}_1} f_k(\zeta' + z_1, \dots, z_n)}{\zeta'} d\zeta'$$

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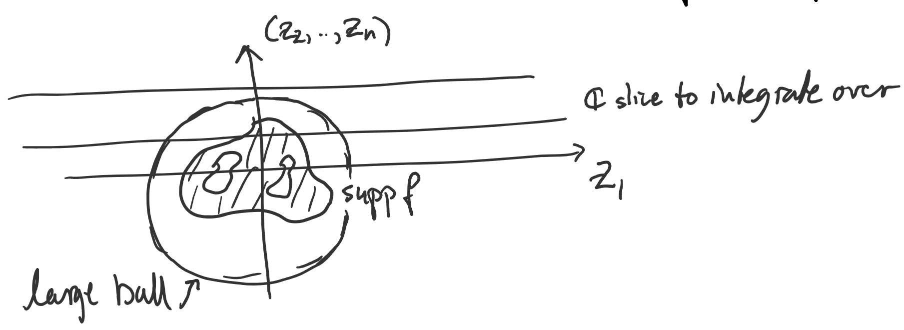
$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial_{\bar{\zeta}} f_k(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \overset{\text{Prop1}}{=} f_k$$

Thus, u solves $\bar{\partial} u = f$. In particular $\bar{\partial} u = 0$ outside $\operatorname{supp} f \subset\subset \mathbb{C}^n$. Since $u \equiv 0$ when $|z_2|^2 + \dots + |z_n|^2 >> 1$, we conclude, by unique continuation, that $u \equiv 0$ outside some ball of radius $>> 1$. Thus, u has compact support. ▨



<u>Remark</u>. The supp. of u will be contained in the union of K and the bdd components of $\mathbb{C}^n \setminus K$. Thus, $u \equiv 0$ in the unbdd component of $\mathbb{C}^n \setminus K$.